An Equivalence Principle for Nonuniform Transmission-Line Directional Couplers

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Abstract—**The analysis of transmission-line directional couplers is formulated in terms of a pair of first-order matrix differential equations. It is shown that for every nonuniform directional coupler that is electrically symmetric, there exists an equivalent pair of dual nonuniform transmission lines. It is also shown that a matched, transmission-line directional coupler having an absolutely continuous characteristic impedance matrix must be symmetric. Restrictions on the terminating impedances and the implications of these restrictions on the realizability of transmission-line couplers are investigated. Finally, the tapered-line magic *T* is treated as an example.**

INTRODUCTION

THE EQUIVALENCE between stepped, transmission-line directional couplers and stepped, transmission-line filters is well known and has been established analytically by several authors [1], [2]. These couplers, which consist of sections of equal-length uniform line, are usually assumed to be matched at each port and to have perfect directivity at all frequencies. The equivalence states that for such structures a two-port transmission-line network can be found which yields as a solution the desired parameters or characteristics of the four-port coupler. The principal purpose of this paper is to show that this equivalence holds for a general class of symmetric, nonuniform directional couplers which includes the stepped coupler as a special case.

It should be said that the existence of such an equivalence is implicit in the work of previous authors [3]–[5]. However, a rigorous justification for its application has not been given. While this may appear unnecessary from a practical point of view, it will be shown that such an approach does lead to some explicit results regarding the realizability of transmission-line directional couplers. Moreover, an analytical treatment of equivalence will provide a basis for the introduction to the coupled-line problem of synthesis techniques recently developed for single, nonuniform lines.

The fact that such an equivalence is possible is suggested by considering the scattering matrix of the general contra-directional coupler depicted in Fig. 1,

$$S(s) = \begin{bmatrix} 0 & S_{12}(s) & S_{13}(s) & 0 \\ S_{12}(s) & 0 & 0 & S_{24}(s) \\ S_{13}(s) & 0 & 0 & S_{34}(s) \\ 0 & S_{24}(s) & S_{34}(s) & 0 \end{bmatrix}, \tag{1}$$

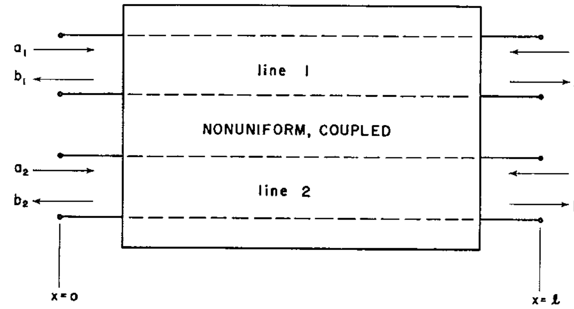

Fig. 1. A four-port transmission-line coupler.

where $s=\sigma+j\omega$ denotes the complex frequency variable. It is well known that for any lossless linear network, $S(s)$ must be unitary. That is, on the real frequency axis, $\bar{S}'(j\omega)S(j\omega)=I$, where the prime denotes the transpose, the bar the complex conjugate, and I the 4×4 unit matrix. This relation is a special case of the para-unitary condition,

$$S'(-s)S(s) = I, \tag{2}$$

which can be derived directly from the Wronskian of the matrix differential equation describing the system.

In the following it will be assumed that the directional coupler possesses side-by-side electrical symmetry, that is, $S_{13}(s)=S_{24}(s)$.[1] In the Appendix it is shown that one has very little freedom in choosing S_{13} and S_{24} independently. In fact, for the class of absolutely continuous coupled lines of equal length S_{13} must equal S_{24}. For directional couplers which are symmetric in this sense (2) reduces to three independent equations that can be combined into

$$[S^{\pm}(-s)]'S^{\pm}(s) = I, \tag{3}$$

where the reduced 2×2 matrices are defined by

$$S^{\pm}(s) = \begin{bmatrix} \pm S_{12}(s) & S_{13}(s) \\ S_{13}(s) & \pm S_{34}(s) \end{bmatrix}. \tag{4}$$

In (3) and hereafter I will be used to denote the 2×2 unit matrix. Equation (4) suggests that there exist two single transmission lines having as their reflection coefficients S_{12} and $-S_{12}$, respectively, and having S_{13} as their transmission coefficient. It will be shown that this conjecture is true for the class of nonuniform couplers considered.

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[1] This definition of symmetry does not necessarily imply physical symmetry of any kind.

The Matrix Line Equations

It will be obvious that the discussion of this section applies equally well to n coupled lines. However, nothing is gained by the added generality, and in the interest of simplicity the discussion will be limited to the case where $n=2$. Denote by the column matrices,

$$e(z) = \begin{bmatrix} e_1(z) \\ e_2(z) \end{bmatrix}$$

$$i(z) = \begin{bmatrix} i_1(z) \\ i_2(z) \end{bmatrix} \tag{5}$$

the voltages and currents, respectively, on the two lines depicted in Fig. 1. The coupled, nonuniform line equations for the lossless case can then be written

$$\frac{de(z)}{dz} = - sL(z)i(z) \tag{6a}$$

$$\frac{di(z)}{dz} = - sC(z)e(z), \tag{6b}$$

where $L(z)$ and $C(z)$ are real, symmetric 2×2 matrices. It will be assumed that the elements of L and C are bounded and integrable functions of z and at each point on the line satisfy conditions consistent with TEM-mode propagation. Thus, from energy considerations both L and C must be positive semidefinite for every z. For reasons which will be apparent shortly it will be assumed that L and C are actually positive definite.

In the case of uniformly coupled, parallel conductors in a homogeneous media the inductance and capacitance matrices in (6a) and (6b) are, of course, independent of z. Moreover, it is known [6] that $LC=CL=\mu\epsilon I$. It is easily shown from Maxwell's equations that this relation as well as (6a) and (6b) still hold when the media is inhomogeneous in the z direction, that is, when $\mu=\mu(z)$ and $\epsilon=\epsilon(z)$. It makes sense to assume, therefore, that in the general nonuniform case in which the cross section of the structure may also vary with z that there still exists a positive function $v(z)$ such that

$$L(z)C(z) = C(z)L(z) = \frac{1}{v^2(z)} I. \tag{7}$$

This condition is consistent with the TEM assumption since it implies that at each point z the propagation on both lines is characterized by the same "local" velocity, $v(z)$. While (7) is difficult to justify in the case where the propagating field is not strictly TEM, it will now be shown that this assumption does lead to the definition of a unique "local" characteristic impedance matrix which reduces to the conventional result for the case of uniform coupled lines. Thus, at each point on the nonuniform structure it is possible to associate a system of coupled, uniform lines having the same inductance and capacitance matrices at that point. As in the case of single lines it would be expected that the error made in this type of approximation would be acceptable for coupled lines having only a moderate degree of nonuniformity.

Introduce the new variable

$$x(z) = \int_0^z \frac{d\xi}{v(\xi)} \,. \tag{8}$$

Noting that z is determined uniquely in terms of x by this relation, we will express the spatial dependence of the various functions as follows: $e[z(x)]=e(x)$, $L[z(x)]=L(x)$, etc. Then,

$$\frac{de(z)}{dz} = \frac{de[z(x)]}{dx}\frac{dx}{dz} = \frac{de(x)}{dx}\frac{1}{v(z)} \,. \tag{9}$$

Since every positive definite matrix has an inverse and a unique positive definite square root, (7) and (9) and a similar equation in $i(x)$ can be used to put (6a) and (6b) in the symmetric form,

$$\frac{de(x)}{dx} = - sZ_0(x)i(x) \tag{10a}$$

$$\frac{di(x)}{dx} = - sZ_0^{-1}(x)e(z), \tag{10b}$$

where the characteristic impedance matrix is defined by

$$Z_0(x) = C(x)^{-1/2}L(x)^{1/2}. \tag{11}$$

From the assumed properties of $L(x)$ and $C(x)$ it can be shown that $Z_0(x)$ is symmetric and positive definite. The proof follows from the well-known result that if two symmetric matrices commute they can be diagonalized by the same orthogonal matrix [7]. It is evident from (7) that (11) can also be written

$$Z_0(x) = \frac{1}{v(x)} C^{-1}(x), \tag{12}$$

so that, given $v(x)$, there is a one-to-one correspondence between $Z_0(x)$ and $C(x)$. The question of whether or not a particular $C(x)$ can be physically realized in the quasi TEM sense previously discussed will be deferred until later.

Properties of Transmission-Line Directional Couplers

The scattering matrix formulation is often the most convenient one to use when discussing directional couplers. This is particularly true of transmission-line couplers if the scattering matrix is partitioned so as to distinguish between input and output ports. No assumption will be made in this section regarding symmetry. It will be assumed initially that the ith port of the coupler is terminated in a uniform, lossless line of characteristic impedance $r_i>0$. Therefore, the scattering matrix is appropriately normalized with respect to the real characteristic impedance matrix,

$$R = \left[\begin{array}{cc|cc} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ \hline 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{array}\right] = \left[\begin{array}{c|c} R_1 & 0 \\ \hline 0 & R_2 \end{array}\right]. \tag{13}$$

If we define the column matrices,

$$\boldsymbol{E}(s) = \begin{bmatrix} e_1(0, s) \\ e_2(0, s) \\ e_1(l, s) \\ e_2(l, s) \end{bmatrix}$$

$$\boldsymbol{I}(s) = \begin{bmatrix} i_1(0, s) \\ i_2(0, s) \\ -i_1(l, s) \\ -i_2(l, s) \end{bmatrix}, \tag{14}$$

then the scattering matrix $S(s)$ satisfies

$$\boldsymbol{b} = S\boldsymbol{a}, \tag{15}$$

where

$$2\boldsymbol{a}(s) = R^{-1/2}\boldsymbol{E}(s) + R^{1/2}\boldsymbol{I}(s)$$

$$2\boldsymbol{b}(s) = R^{-1/2}\boldsymbol{E}(s) - R^{1/2}\boldsymbol{I}(s) \tag{16}$$

and, referring to Fig. 1, $\boldsymbol{a}=(a_1a_2a_3a_4)'$ and $\boldsymbol{b}=(b_1b_2b_3b_4)'$.

The fundamental matrix solution to the system is now defined as the 2×2 matrices,

$$E(x, s) = \begin{bmatrix} e_1^{(1)}(x, s) & e_1^{(2)}(x, s) \\ e_2^{(1)}(x, s) & e_2^{(2)}(x, s) \end{bmatrix}$$

$$I(x, s) = \begin{bmatrix} i_1^{(1)}(x, s) & i_1^{(2)}(x, s) \\ i_2^{(1)}(x, s) & i_2^{(2)}(x, s) \end{bmatrix} \tag{17}$$

in which the corresponding columns are independent vector solutions of (10a) and (10b) satisfying the boundary conditions, $E(l, s)=R_2^{1/2}$ and $I(l, s)=R_2^{-1/2}$. The dependence of the solution on the frequency variable s is now explicitly indicated. Clearly we can write

$$\frac{dE(x, s)}{dx} = -sZ_0(x)I(x, s) \tag{18a}$$

$$\frac{dI(x, s)}{dx} = -sZ_0^{-1}(x)E(x, s). \tag{18b}$$

Associated with the two vector solutions are the 4×2 parameter matrices,

$$\begin{bmatrix} A_\alpha \\ \hline A_\beta \end{bmatrix} = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} \\ a_2^{(1)} & a_2^{(2)} \\ \hline a_3^{(1)} & a_3^{(2)} \\ a_4^{(1)} & a_4^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} B_\alpha \\ \hline B_\beta \end{bmatrix} = \begin{bmatrix} b_1^{(1)} & b_1^{(2)} \\ b_2^{(1)} & b_2^{(2)} \\ \hline b_3^{(1)} & b_3^{(2)} \\ b_4^{(1)} & b_4^{(2)} \end{bmatrix}. \tag{19}$$

Evidently,

$$\begin{bmatrix} B_\alpha \\ \hline B_\beta \end{bmatrix} = S \begin{bmatrix} A_\alpha \\ \hline A_\beta \end{bmatrix}, \tag{20}$$

where, for a matched directional coupler,

$$S = \left[\begin{array}{cc|cc} 0 & S_{12} & S_{13} & 0 \\ S_{12} & 0 & 0 & S_{24} \\ \hline S_{13} & 0 & 0 & S_{34} \\ 0 & S_{24} & S_{34} & 0 \end{array}\right] = \left[\begin{array}{c|c} S_{\alpha\alpha} & S_{\alpha\beta} \\ \hline S_{\beta\alpha} & S_{\beta\beta} \end{array}\right]. \tag{21}$$

The partitioning indicated in (21) allows us to treat the four-port coupler as a two-port network. This expedient is useful as long as the partitioned matrices are nonsingular, which is guaranteed by our choice of the fundamental matrix solution. From these equations it follows that

$$2A_\alpha = R_1^{-1/2}E(0, s) + R_1^{1/2}I(0, s) \tag{22a}$$

$$2B_\alpha = R_1^{-1/2}E(0, s) - R_1^{1/2}I(0, s) \tag{22b}$$

$$2A_\beta = R_2^{-1/2}E(l, s) - R_2^{1/2}I(l, s) \tag{22c}$$

$$2B_\beta = R_2^{-1/2}E(l, s) + R_2^{1/2}I(l, s). \tag{22d}$$

Inserting the boundary conditions in (22c) and (22d) we find that $A_\beta=0$ and $B_\beta=I$. It follows from (20) through (22b) that

$$E(0, s) = R_1^{1/2}(A_\alpha + B_\alpha) = R_1^{1/2}(I + S_{\alpha\alpha})S_{\beta\alpha}^{-1} \tag{23}$$

and

$$I(0, s) = R_1^{-1/2}(A_\alpha - B_\alpha) = R_1^{-1/2}(I - S_{\alpha\alpha})S_{\beta\alpha}^{-1}. \tag{24}$$

A basic premise in transmission-line theory is the uniqueness of voltage and current. Mathematically, this property can be derived by expressing (18a) and (18b) in the form of a matrix Volterra integral equation. In this way it can be shown that $E(x, s)$ and $I(x, s)$ are continuous functions of x for all s and that either function is uniquely determined by the values of E and I at one point, say $x=l$. This property is subject only to the condition that $Z_0(x)$ be integrable. In the practical situation where the elements of $Z_0(x)$ are sectionally continuous this requirement is certainly satisfied. A second property which can be deduced from the integral representation is that $E(x, s)$ and $I(x, s)$ are entire functions of s for all x, $0\leq x\leq l$. We now wish to examine the implications of uniqueness on the terminal impedances of the coupler. Consider (18a) when $s=0$. The solution is $E(x, 0)=C$, where C is a constant 2×2 matrix.

But from the boundary conditions for the fundamental solution, $C=R_2^{1/2}$, and it is concluded that $E(0, 0)=R_2^{1/2}$ and $I(0, 0)=R_2^{-1/2}$. Subtracting (24) from (23) and employing (21) we obtain for $s=0$,

$$R_1^{-1/2}R_2^{1/2} - R_1^{1/2}R_2^{-1/2} = 2S_{\alpha\alpha}(0)S_{\beta\alpha}^{-1}(0) = \begin{bmatrix} 0 & \dfrac{2S_{12}(0)}{S_{24}(0)} \\ \dfrac{2S_{12}(0)}{S_{13}(0)} & 0 \end{bmatrix}. \quad (25)$$

Since R_1 and R_2 are diagonal, it follows that $R_1=R_2$. In other words, to be a directional coupler the lines must be loaded in a symmetric fashion so that $r_1=r_3$ and $r_2=r_4$. It also follows from (25) that $S_{12}(0)=0$ and from (23), $S_{13}(0)=S_{24}(0)=1$.

Analysis

In view of the above constraint on the terminal impedances, the following transformation is suggested:

$$\hat{E}(x, s) = R^{-1/2}E(x, s)$$
$$\hat{I}(x, s) = R_1^{1/2}I(x, s) \cdot \quad (26)$$

Substituting (26) in (18a) and (18b), the matrix line equations become

$$\frac{d\hat{E}(x, s)}{dx} = -sR_0(x)\hat{I}(x, s) \quad (27a)$$

$$\frac{d\hat{I}(x, s)}{dx} = -sR_0^{-1}(x)\hat{E}(x, s), \quad (27b)$$

where

$$R_0 = R_1^{-1/2}Z_0(x)R_1^{-1/2} \cdot \quad (28)$$

It will now be assumed that the coupler possesses electrical symmetry in the sense that $S_{13}=S_{24}$. The boundary conditions for the normalized voltage and current at $x=0$ and $x=l$ can then be written,

$$\hat{E}(0, s) = \begin{bmatrix} 1/S_{13} & S_{12}/S_{13} \\ S_{12}/S_{13} & 1/S_{13} \end{bmatrix}$$

$$\hat{I}(0, s) = \begin{bmatrix} 1/S_{13} & -S_{12}/S_{13} \\ -S_{12}/S_{13} & 1/S_{13} \end{bmatrix} \quad (29)$$

and $\hat{E}(l, s)=\hat{I}(l, s)=I$. The "double symmetry" displayed in the boundary conditions plays a key role in the following analysis. We shall consider the class of solutions of (27a) and (27b) which are doubly symmetric for all x in the interval $0\leq x\leq l$; that is, solutions for which $\hat{e}_1^{(1)}(x, s)=\hat{e}_2^{(2)}(x, s)$ and $\hat{e}_2^{(1)}(x, s)=\hat{e}_1^{(2)}(x, s)$, and similarly for the components of the current matrix. The existence of such solutions will be shown to follow from the existence of the solutions for two single-line problems. Under this assumption it is concluded from (27a) that $R_0(x)$ must also be doubly symmetric for all x. That is, $R_0(x)$ has the form,

$$R_0(x) = \begin{bmatrix} R_{11}(x) & R_{12}(x) \\ R_{12}(x) & R_{11}(x) \end{bmatrix}. \quad (30)$$

Adding and subtracting columns of (27a) and (27b) gives

$$\frac{d}{dx}[\hat{e}_1^{(1)}(x, s) \pm \hat{e}_1^{(2)}(x, s)] = -s[R_{11}(x) \pm R_{12}(x)][\hat{\imath}_1^{(1)}(x, s) \pm \hat{\imath}_1^{(2)}(x, s)] \quad (31a)$$

$$\frac{d}{dx}[\hat{\imath}_1^{(1)}(x, s) \pm \hat{\imath}_1^{(2)}(x, s)] = -s[G_{11}(x) \pm G_{12}(x)][\hat{e}_1^{(1)}(x, s) \pm \hat{e}_1^{(2)}(x, s)], \quad (31b)$$

where $G_{11}(x)$ and $G_{12}(x)$ are the corresponding elements of $G_0(x)=R_0^{-1}(x)$. Thus this procedure, which is analogous to the familiar even- and odd-mode analysis of physically symmetric microwave structures, has resulted in the separation of the matrix transmission-line equations into two scalar equations. Stated in other terms, the condition of double symmetry makes it possible to diagonalize simultaneously both (27a) and (27b) with a constant, orthogonal matrix. After making the definitions, $E^{\pm}(x, s)=\hat{e}_1^{(1)}(x, s)\pm\hat{e}_1^{(2)}(x, s)$, $I^{\pm}(x, s)=\hat{\imath}_1^{(1)}(x, s)\pm\hat{\imath}_1^{(2)}(x, s)$, and $R^{\pm}(x)=R_{11}(x)\pm R_{12}(x)$ and noting that

$$G^{\pm}(x) \equiv G_{11}(x) \pm G_{12}(x) = 1/R^{\pm}(x), \quad (32)$$

(31a) and (31b) become

$$\frac{dE^{\pm}(x, s)}{dx} = -sR^{\pm}(x)I^{\pm}(x, s) \quad (33a)$$

$$\frac{dI^{\pm}(x, s)}{dx} = -s1/R^{\pm}(x)E^{\pm}(x, s). \quad (33b)$$

The boundary conditions for these normalized equations are

$$E^{\pm}(0, s) = \frac{1\pm S_{12}(s)}{S_{13}(s)}$$
$$I^{\pm}(0, s) = \frac{1 \mp S_{12}(s)}{S_{13}(s)} \quad (34a)$$
$$E^{\pm}(l, s) = I^{\pm}(l, s) = 1. \quad (34b)$$

It is evident that (33) and (34) describe two nonuniform lines each terminated in one ohm and possessing the scattering parameters $S_{11}^{\pm}(s)=\pm S_{12}(s)$ and $S_{12}^{\pm}(s)=S_{13}(s)$ normalized to one ohm. From (28) and the fact that $Z_0(x)$ is positive definite it follows that $R_{11}>|R_{12}|$. Consequently, $R^{\pm}(x)>0$, $0\leq x\leq l$, and the sum and difference lines are always realizable. This completes the proof of the conjecture made in the Introduction.

It will now be shown that the sum and difference transmission-line problems are not independent; in fact, one line is the dual of the other. The proof of this statement follows from the uniqueness of the synthesis problem for nonuniform lines. The question of uniqueness and general realizability for continuously nonuniform lines has only recently begun to receive attention. Wohlers [8] has shown that for lossless lines possessing a local characteristic impedance $R(x)$ that is twice continuously differentiable, the reflection coefficient $S_{11}(s)$ uniquely determines $R(x)$. Heim and Sharpe [9] have obtained the same result under the more general condition that $R(x)$ be only absolutely continuous. The uniqueness of the synthesis problem for general discontinuous lines has yet to be proved, although there are strong heuristic reasons for assuming that the realizability of $S_{11}(s)$ always guarantees the uniqueness of $R(x)$. In the familiar case of stepped, uniform lines of equal length it is evident from the method of extracting sections that the synthesis problem has a unique solution [10].[2] It is also known that $S_{11}(s)$ uniquely determines the other elements of the scattering matrix if R is absolutely, sectionally continuous[3] and l is known. This implies that either $E(0, s)$ or $I(0, s)$ is sufficient to prescribe the synthesis of a lossless line terminated in a real impedance.

Assume that the synthesis problem for the sum and difference lines associated with (33) has a unique solution. That is, given a realizable $E^+(0, s)$, $R^+(x)$ is uniquely determined and similarly for $E^-(0, s)$ and $R^-(x)$. But from (34a) $E^{\pm}(0, s) = I^{\mp}(0, s)$. It follows that[4]

$$R^+(x) = 1/R^-(x), \qquad 0 \le x \le l. \tag{35}$$

Furthermore, the voltage on the sum line is equal to the current on the difference line, and vice versa, for every x. Thus, the two lines are complete duals. Since $R^+(x)$ corresponds to the "even-mode impedance" Z_{0e}, and $R^-(x)$ to the "odd-mode impedance" Z_{00}, (35) is a generalization of the well-known condition, $Z_{0e}Z_{00} = 1$, which is necessary for the equivalence of stepped lines and stepped directional couplers.

It has been shown that the realizability of a symmetric coupler guarantees the realizability of the sum and difference lines. Unfortunately, the converse is not true. This state of affairs results from the fact that the positive definite character of $C(x)$ is not sufficient to guarantee at each value of x the existence of a uniform multiconductor line having the capacitance matrix $C(x)$. It is known that a necessary condition for realizability is that $C(x)$ be hyperdominant [6]. That is, the elements of $C(x)$ must satisfy for all x, $0 \le x \le l$,

$$\sum_{j=1}^{2} C_{ij}(x) \ge 0, \qquad i = 1, 2 \tag{36}$$

and

$$C_{12}(x) = C_{21}(x) \le 0. \tag{37}$$

These conditions cannot be inferred from the positive definite character of $C(x)$. Since (36) and (37) imply that $C(x)$ is positive semidefinite, however, it is possible that the hyperdominant restriction is sufficient as well as necessary, although to the author's knowledge this has not been proved. In any event (36) and (37) do impose restrictions on the characteristic impedance of the sum and difference lines which may complicate the synthesis problem when unequal terminating impedances are specified.

In order to determine these restrictions it will be convenient to deal with the characteristic admittance $Y_0(x) = Z_0^{-1}(x)$ and terminating conductances, $g_1 = 1/r_1$, and $g_2 = 1/r_2$. From (12) and (28),

$$C(x) = \frac{1}{v(x)} \begin{bmatrix} \sqrt{g_1} & 0 \\ 0 & \sqrt{g_2} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{11} \end{bmatrix} \begin{bmatrix} \sqrt{g_1} & 0 \\ 0 & \sqrt{g_2} \end{bmatrix}$$

$$= \frac{1}{v(x)} \begin{bmatrix} g_1 G_{11} & \sqrt{g_1 g_2}\, G_{12} \\ \sqrt{g_1 g_2}\, G_{12} & g_2 G_{11} \end{bmatrix}. \tag{38}$$

Condition (36) in conjunction with (32) and (35) leads to the right-hand inequality of

$$1 \ge G^+(x) \ge \sqrt{\left| \frac{\sqrt{g_1} - \sqrt{g_2}}{\sqrt{g_1} + \sqrt{g_2}} \right|} \tag{39}$$

while (37) leads to the left-hand inequality. In summary, a symmetric transmission-line directional coupler is realizable only if the corresponding characteristic admittance $G^+(x)$ of the equivalent line satisfies (39) for all x.

Unequal Terminating Impedances

In order to investigate the role of the impedance transformation given in (26) consider the familiar single-section stepped coupler with line (1) terminated at ports 1 and 3 in a conductance g_1, and line (2) terminated at ports 2 and 4 in a conductance g_2. This example will also bring out the importance of the uniqueness property. The appropriate sum and difference problems are depicted in Fig. 2. The reflection coefficients $S_{11}{}^{\pm}$ are readily found from the solutions to (33a) and (33b). The result is

$$S_{11}{}^{\pm} = \pm S_{12} = \frac{jk^{\pm} \sin \omega l}{\sqrt{1 - (k^{\pm})^2} \cos \omega l + j \sin \omega l}, \tag{40}$$

where

$$k^{\pm} = \frac{(R^{\pm})^2 - 1}{(R^{\pm})^2 + 1}. \tag{41}$$

A unique solution for S_{12} will be obtained only if $k^- = -k^+$. It follows that

$$R^+R^- = G^+G^- = 1 \tag{42}$$

and

$$k^+ = \frac{R^+ - R^-}{R^+ + R^-} = \frac{G^- - G^+}{G^- + G^+}. \tag{43}$$

[2] This statement of uniqueness does not conflict with the multiplicity of solutions to the synthesis problem when the insertion loss rather than $S_{11}(s)$ is specified [11].

[3] $R(x)$ is absolutely continuous except for a finite number of discontinuities.

[4] *Added in proof:* The sufficiency of this condition for physically symmetric directional couplers has recently been established by S. Yamamoto, T. Azakami, and K. Itakura, ("Coupled nonuniform transmission line and its applications," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-15, pp. 220–231, April 1967).

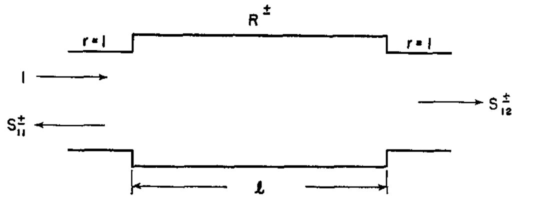

Fig. 2. Equivalent lines for a single-section directional coupler.

It remains to relate these parameters to the familiar even- and odd-mode characteristic admittances. Using Cristal's notation [12] the characteristic admittance matrix will be written

$$Y_0 = \begin{bmatrix} \frac{Y_{0e_1} + Y_{00_1}}{2} & \frac{Y_{0e_1} - Y_{00_1}}{2} \\ \frac{Y_{0e_1} - Y_{00_1}}{2} & \frac{Y_{0e_2} + Y_{00_2}}{2} \end{bmatrix} \equiv \begin{bmatrix} A & -D \\ -D & B \end{bmatrix}. \tag{44}$$

$$\frac{S_{12}(j\omega)}{S_{13}(j\omega)} \cong e^{j\omega l} \frac{K(\Gamma_2 e^{-2j\omega l} - \Gamma_1) + \int_0^l [e^{-2j\omega\xi} - \Gamma_1\Gamma_2 e^{-2j\omega(l-\xi)}]P(\xi)d\xi}{(1+\Gamma_2)(1-\Gamma_1)}, \tag{47}$$

From (28)

$$G_0 = R_0^{-1} = \begin{bmatrix} A/g_1 & -D/\sqrt{g_1 g_2} \\ -D/\sqrt{g_1 g_2} & B/g_2 \end{bmatrix}. \tag{45}$$

G_0, and hence R_0, will be doubly symmetric only if $A/g_1 = B/g_2$, which is Cristal's matching condition. Equations (32) and (42) lead to $AB - D^2 = g_1 g_2$, which is Cristal's condition for infinite directivity. Also, from (43), the coupling coefficient is $k^+ = D/\sqrt{AB}$. Other formulas giving g_1 and g_2 in terms of the components of the L and C matrices have been given elsewhere [13] and will not be repeated here. It is clear from the example that the so-called nonsymmetric coupler is not basically different from the more common coupler for which $g_1 = g_2$. What is important is that the coupler possess electrical symmetry as defined previously in terms of the scattering parameters. Finally, it should be noted that if $S_{12}(s)$ or $G^+(x)$ is given, the terminating conductances g_1 and g_2 cannot be chosen arbitrarily.

Analysis of Nonuniform Directional Couplers

The principle advantage offered by continuously tapered couplers lies in the possibility of obtaining extremely wide coupling bandwidths. This contrasts with the periodic coupling characteristic of stepped directional couplers having equal-length sections. Theoretically, the coupling bandwidth will be infinite if $R^+(x)$ can be made discontinuous. However, the practical difficulty of realizing such a step in impedance leaves this ultimate bandwidth open to question. A more recognizable advantage seems to lie in the use of continuous nonuniform couplers to eliminate (in a quasi-TEM sense) reactive discontinuities altogether, thus removing a major deterrent to the achievement of high directivity.

It is often convenient to specify the properties of directional couplers in terms of the power division factor

$$\eta = \frac{|S_{12}|^2}{|S_{13}|^2}. \tag{46}$$

It can be shown that in the case of stepped couplers η is an even polynomial in $\cos \omega l$, where l is the normalized length of each section [3]. In the continuous case this parameter can be readily approximated using techniques developed for the analysis of single nonuniform lines. In order to accommodate possible discontinuities in $R^+(x)$ at $x=0$ and $x=l$, we will employ the formulation given by Youla [14]. Noting the boundary conditions given in (34) for the equivalent transmission-line problem, we obtain for the first-order approximation,

where

$$P(\xi) = 1/2 \frac{d}{d\xi} \ln R^+(\xi) \tag{48a}$$

$$\Gamma_1 = \frac{1 - R^+(0)}{1 + R^+(0)}$$

$$\Gamma_2 = \frac{1 - R^+(l)}{1 + R^+(l)} \tag{48b}$$

and

$$K = 1 - \ln \sqrt{R^+(l)/R^+(0)}. \tag{48c}$$

As an illustration in the application of this expression it will be instructive to consider the tapered-line magic T [15]. For reasons of simplicity we will assume that $R^+(x)$ and $R^-(x)$ have an exponential taper in the nonuniform region rather than the Chebychev taper originally employed. The variation of $R^+(x)$ is depicted by the solid line in Fig. 3. Even though the output terminals are located at $x=2l$, better accuracy will be obtained if (47) is applied only to the nonuniform section, $0 \leq x \leq l$, and a correction for the phase shift introduced by the uniform section is made afterward. Setting $\kappa = 1/R^+(l) = e^{-2\alpha l}$ and noting that $\Gamma_1 = 0$ and $\Gamma_2 = (\kappa - 1)/(\kappa + 1)$, we find that

$$\frac{S_{12}(j\omega)}{S_{13}(j\omega)} \cong \frac{\kappa + 1}{2\kappa}\left[(1 - \alpha l)\left(\frac{\kappa - 1}{\kappa + 1}\right) + \alpha l e^{j\omega l} \frac{\sin \omega l}{\omega l}\right]. \tag{49}$$

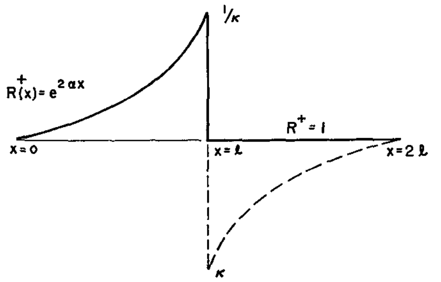

Fig. 3. Impedance variation of the equivalent line for a tapered-line magic *T*.

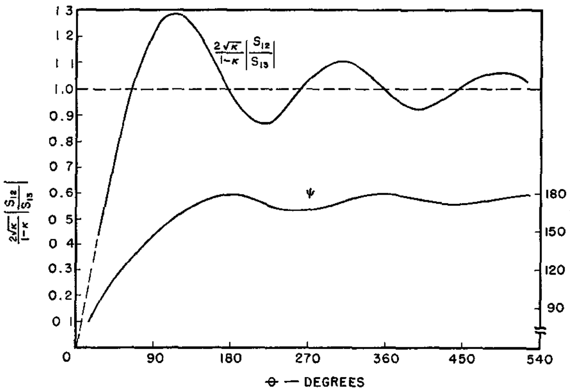

Fig. 4. Amplitude and phase variation of the power division factor for a tapered-line magic *T*.

This result, while reasonably accurate at intermediate frequencies, is not correct at either $\omega=0$ or $\omega=\infty$. The correct asymptotic behavior at infinity, as given by DuHamel and Armstrong, [15] can be obtained by making the substitutions, $\alpha l \cong (1-\kappa)/(1+\kappa)$ and $1-\alpha l \cong \sqrt{\kappa}$. This can be justified when $\alpha l \ll 1$, since these approximations are of the same order as the approximations inherent in the derivation of (47). The final result is

$$\frac{S_{12}(j\omega)}{S_{13}(j\omega)} = \sqrt{\eta}\, e^{j\psi} = \frac{\kappa-1}{2\sqrt{\kappa}}\left[1 - \frac{e^{j\omega l}}{\sqrt{\kappa}} \frac{\sin \omega l}{\omega l}\right]. \tag{50}$$

The magnitude and phase of this expression are plotted in Fig. 4 versus $\theta=\omega l$ for the case where $\kappa=0.818$, corresponding to a 20 dB asymptotic coupling coefficient. Equation (47) provides a convenient formula for evaluating other tapers for directional couplers. For example, it can be shown by employing a double taper such as that suggested by the dotted line in Fig. 3, that the variation of the phase $\psi(\omega)$ can be substantially reduced. Unfortunately, because this characteristic violates (39), it does not lead to a realizable transmission-line coupler.[5] As in the case of stepped couplers, ψ can be made to approximate 90 degrees by employing an $R^+(x)$ characteristic which is symmetric about $x=l$ instead of antisymmetric as in the previous case. However, infinite coupling bandwidth is then no longer possible since, in the absence of a discontinuity in $R^+(x)$, η approaches zero for large frequencies. In spite of this restriction Tresselt [5] has recently reported obtaining a 7 to 1 bandwidth with such a design.

[5] This type of coupler was first discussed by Oliver [4]. He observed that it could be realized by transposing conductors, an operation which is ruled out in the present theory.

Conclusion

The important implication of the equivalence principle presented here is that whenever an exact solution to the analysis or synthesis problem for nonuniform transmission lines can be found, an exact solution to the corresponding directional coupler problem is also available, subject to the realizability conditions stated earlier. Thus, nonuniform directional couplers can be designed without necessarily making the usual assumption that the coupling is small or that the lines are only slightly nonuniform. Although there is little in the literature on exact methods of nonuniform transmission-line synthesis, there is increasing interest in the subject and hopeful signs that important advances will soon be forthcoming. In what may be the first practical solution to this problem, Heim [16] has recently devised an exact technique for constructing the characteristic impedance of a class of absolutely continuous lines. However, much work remains to be done, particularly on the approximation problem and on the realizability problem encountered here where the characteristic impedance is bounded from above and below.

Appendix

In this section a proof will be given for the following theorem: A matched, transmission-line directional coupler having an absolutely continuous characteristic impedance matrix must be symmetric; that is, $S_{13}(s)=S_{24}(s)$. It is understood that the two lines making up the coupler are of equal length.

It can be shown from the properties of the solution that $S_{13}(s)$ and $S_{24}(s)$ are meromorphic functions of s and analytic in the half plane, $\sigma \geq 0$. Therefore, S_{13}/S_{24} is also a meromorphic function, and we can write in general,

$$\frac{S_{13}(s)}{S_{24}(s)} = \frac{F(s)}{G(s)}, \tag{51}$$

where $F(s)$ and $G(s)$ are entire functions of s. The phase law,

$$S_{13}(s)S_{13}(-s) = S_{24}(s)S_{24}(-s), \tag{52}$$

which can be deduced from (2), implies that $G(s)$ and $F(-s)$ have the same zeros. It follows from Hadamard's factorization theorem [17] that

$$S_{24}(s) = S_{13}(s)\frac{F(-s)}{F(s)} e^{f(s)}, \tag{53}$$

where $f(s)$ is a polynomial of finite degree. We have seen that in order for a matched directional coupler to exhibit perfect directivity at all frequencies it is necessary that $S_{13}(0) = S_{24}(0) = 1$. From this condition we conclude that $f(0)=0$. It will now be shown that $F(s)$ can have no zeros. From (23) and (24),

$$S_{\beta\alpha}^{-1} = \begin{bmatrix} 1/S_{13} & 0 \\ 0 & 1/S_{24} \end{bmatrix}$$
$$= \tfrac{1}{2}[R_1^{-1/2}E(0, s) + R_1^{1/2}I(0, s)]$$
$$= \tfrac{1}{2}[\hat{E}(0, s) + \hat{I}(0, s)]. \quad (54)$$

Since the right side is an entire function of s, neither $S_{13}(s)$ or $S_{24}(s)$ can be zero for any finite value of s. Referring to (53) any zeros of $F(-s)$ which are canceled by poles of $S_{13}(s)$ produce poles of $S_{24}(s)$ in the right half plane. Consequently, $F(s)$ can have no zeros, and S_{13} and S_{24} differ at most by an exponential factor.

It remains to evaluate the asymptotic behavior of $S_{13}(s)$ and $S_{24}(s)$ as $s \to \infty$. In general the existence of a fundamental solution satisfying $\hat{E}(l, s) = \hat{I}(l, s) = I$ does not permit us to infer anything about the asymptotic behavior of $\hat{E}(x, s)$ and $\hat{I}(x, s)$ at $x=0$. However, it can be shown by extending previous results for the single line [9] to the matrix case that if the elements of $Z_0(x)$ are absolutely continuous, then there exists a fundamental solution to (27) such that for all x, $0 \leq x \leq l$,

$$\hat{E}(x, s) \to \begin{bmatrix} e^{s(l-x)} & 0 \\ 0 & e^{s(l-x)} \end{bmatrix} \quad (55)$$

and

$$\hat{I}(x, s) \to \begin{bmatrix} e^{s(l-x)} & 0 \\ 0 & e^{s(l-x)} \end{bmatrix} \quad (56)$$

uniformly as $s \to \infty$ in the half plane, $\sigma \geq 0$. Since the solution is unique, it follows from (54) that $S_{13}(s)$ and $S_{24}(s)$ have the same asymptotic behavior and are, therefore, equal.

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